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LETTER TO THE EDITOR

Towards new solutions of the general Hurwitz problem

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Abstract. A formulation of the generalized Hurwitz problem is given. We show that the solution of this problem is equivalent to exhibiting a matrix extending Yiu's intercalate matrix consistently signed. We give examples of explicit solutions for such a generalized Hurwitz problem in low dimensions.

The aim of this letter is to investigate solutions of the so-called 'generalized Hurwitz problem'. This problem can be formulated as follows. We are searching for three integers r, s, n and three diagonal matrices N_r, N_s and N_n (respectively $r\times r$, $s\times s$, $n\times n$), whose entries are ± 1 only, such that

$$Z^T N_n Z = (X^T N_r X)(Y^T N_s Y)$$

for all X and Y which are real vectors given by $X^T = (x_1, \ldots, x_r)$ and $Y^T = (y_1, \ldots, y_s)$. Furthermore, the real vector Z defined by $Z^T = (z_1, \ldots, z_n)$ is characterized by the following equations,

$$z_k = \sum_{i=1}^{r} \sum_{j=1}^{s} a_{ij}^{(k)} x_i y_j$$

where $a_{ij}^{(k)}$ belongs to the set $\{0, +1, -1\}$.

A direct computation shows that (i, j) is associated to only *one* k which therefore will be deleted. If such Z exists, we say that we have found the solution of the 'generalized Hurwitz problem' $[(r, s, n); (N_r; N_s, N_n)]$.

1. Solutions of the problem $[(n, n, n); (I_n, I_n, I_n)]$, where I_n denotes the $n\times n$ identity matrix, dates back to Hurwitz in 1898 [1]. He showed that $n = 1, 2, 4$ and 8, and these solutions are, in fact, connected to the multiplication rules defined on the fields of real, complex, quaternionic numbers and on the ring of octonions.

2. Solutions of the problem $[(r, n, n); [I_r, I_n, I_n)]$ were found independently by Radon and Hurwitz [2]. They proved that r has to satisfy the following inequality: $r \leqslant \rho(n)$, where ρ is the so-called 'Hurwitz-radon function' [3]. These solutions are related to purely real, antisymmetric representations of Clifford algebras $C(0, r-1)$ [4].

3. Solutions of the problem $[(r, n, n); (N_r, N_n, N_n')]$ where, N_r, N_n and N_n' are not necessarily identity matrices, were nicely constructed by Lawrynowicz and Rembielinski [5] using the classification of Clifford algebras given by Atiyah *et al* [6].

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This set of solutions was derived and completed by Randriamihamison [7] in the framework of the spinor theory developed by Chevalley and Crumeyrolle [8].

4. When r, s, n are not equal, solutions of the problems $[(r, s, n); (I_r, I_s, I_n)]$ are not known in general. Adem [9], Lam [10] and recently Yiu [11] have derived solutions of such problems for some particular sets of triplets (r, s, n) using sophisticated techniques. The highly non-trivial progress performed by Yiu was to translate the problems $[(r, s, n); [I_r, I_s, I_n)]$ into a combinatorial language using an 'intercalate matrix consistently signed' [12].

In this letter, we give the generalization of Yiu's work for the problem $[(r, s, n); (N_r, N_s, N_n)]$ where N_r, N_s and N_n are not necessarily identity matrices. We then give some explicit examples.

Following Yiu [12] we define M to be an intercalate matrix of type (r, s, n) if and only if it a $r \times s$ matrix with n distinct entries $c_1, c_2, \ldots, c_n$ (called 'colours') satisfying the following conditions:

(i) The colours along each row (respectively, column) are distinct
(ii) If $M_{ij} = M_{i'j'}$ then $M_{ij'} = M_{i'j}$.

Now we introduce a new concept generalizing the consistent signature of Yiu. We say that an intercalate $r \times s$ matrix M is (N_r, N_s)-consistently signed if and only if the following conditions hold:

(iii) each entry M_{ij} is endowed with a sign $a_{ij} = \pm 1$
(iv) $a_{ij}a_{ij'}a_{i'j}a_{i'j'} = -1$ if $(N_r)_i i = (N_r)_{i'i'}$ and $(N_s)_{jj} = (N_s)_{j'j'}$
$= +1$ if $(N_r)_{ii} = -(N_r)_{i'i'}$ and $(N_s)_{jj} = -(N_s)_{j'j'}$.

Such a matrix can be written $A \circ M$ where $A = (a_{ij})$ and where $\circ$ denotes the Hadamard matrix product (component by component). Then, extending the proof of Yiu, it is straightforward to obtain the following theorem (manuscript in preparation).

Theorem. Let $A \circ M$ be an $r \times s$ intercalate matrix (N_r, N_s)-consistently signed. Then

$$z_k = \sum_{M_{ij} = c_k} a_{ij} x_i y_j \qquad k = 1, \ldots, n$$

is a solution of the problem $[(r, s, n); (N_r, N_s, N_n)]$ if and only if N_n is defined by the following condition

$$(N_n)_{kk} = (N_r)_{ii}(N_s)_{jj} \qquad \text{when } M_{ij} = c_k.$$

The solution of this problem can be written using a $n \times r$ matrix $H(Y)$ such that $Z = H(Y)X$. We then immediately check the following corollary.

Corollary. The matrix $H(Y)$ associated with the solution of a $[(r, s, n); (Nr, Ns, Nn)]$ problem satisfies the following equation:

$$H(Y)^T N_n H(Y) = (Y^T N_s Y) N_r.$$

Let us denote p_n and q_n (respectively, p_r and q_r) the number of entries equal to $+1$ and -1 in the matrix N_n (respectively, N_r). If we fix the vector Y such that $Y^T N_s Y = 1$, the equation $Z = H(Y)X$ defines a map f_{nr} between pseudo-Riemannian symmetric spaces. More precisely, if $X^T N_r X = 1$, we obtain

$$f_{nr}: SO(p_r, q_r)/SO(p_r - 1, q_r) \to SO(p_n, q_n)/SO(p_n - 1, q_n)$$

which can be used to generalize what we called 'Hurwitz transformations' in a previous work [13].

Let us now give some explicit examples of solutions for three problems $[(r, s, n); (Nr, Ns, Nn)]$.

(a) $r=s=n=2$ and $N_r=N_s=N_n=N_2$

$$N2=\begin{matrix} +1 & 0 \\ 0 & -1 \end{matrix}=\mathrm{diag}(+,-)$$

$$A\circ M=\begin{matrix} +c_1 & +c_2 \\ +c_2 & +c_1 \end{matrix} \qquad H(Y)=\begin{matrix} y_1 & y_2 \\ y_2 & y_1 \end{matrix}.$$

This well known example is related to the multiplication rule of hyperbolic complex (or double) numbers.

(b) $r=3, s=6, n=8$

$N_3=\mathrm{diag}(+,-,-)$
$N_6=\mathrm{diag}(+,-,-,+,+,-)$
$N_8=\mathrm{diag}(+,-,-,+,+,-,-,+)$

$$A\circ M=\begin{matrix} +c_1 & +c_2 & +c_3 & +c_4 & +c_5 & +c_7 \\ +c_2 & +c_1 & -c_4 & -c_3 & +c_6 & +c_8 \\ +c_3 & +c_4 & +c_1 & +c_2 & +c_7 & +c_5 \end{matrix}$$

$$H(Y)=\begin{matrix} y_1 & y_2 & y_3 \\ y_2 & y_1 & y_4 \\ y_3 & -y_4 & y_1 \\ y_4 & -y_3 & y_2 \\ y_5 & 0 & y_6 \\ 0 & y_5 & 0 \\ y_6 & 0 & y_5 \\ 0 & y_6 & 0 \end{matrix}.$$

This solution can be derived from the multiplication table of split octonions [14] which generalize the usual Cayley–Graves octonions.

(c) $r=10, s=10, n=16$ and $N_r=N_s$

$N_{10}=\mathrm{diag}(+,-,-,+,-,+,+,-,+,-)$
$N_{16}=\mathrm{diag}\,(+,-,-,+,-,+,+,-,+,-,-,+,-,+,+,-)$

$$A\circ M=\begin{matrix} +c_1 & +c_2 & +c & +c_4 & +c_5 & +c_6 & +c_7 & +c_8 & +c_9 & +c_{10} \\ +c_2 & +c_1 & +c_4 & +c_3 & +c_6 & +c_5 & -c_8 & -c_7 & +c_{10} & +c_9 \\ +c_3 & -c_4 & +c_1 & -c_2 & +c_7 & +c_8 & +c_5 & +c_6 & +c_{11} & +c_{12} \\ +c_4 & -c_3 & +c_2 & -c_1 & +c_8 & +c_7 & -c_6 & -c_5 & +c_{12} & +c_{11} \\ +c_5 & -c_6 & -c_7 & -c_8 & +c_1 & -c_2 & -c_3 & -c_4 & +c_{13} & +c_{14} \\ +c_6 & -c_5 & -c_8 & -c_7 & +c_2 & -c_1 & +c_4 & +c_3 & +c_{14} & +c_{13} \\ +c_7 & +c_8 & -c_5 & +c_6 & +c_3 & -c_4 & -c_1 & -c_2 & +c_{15} & +c_{16} \\ +c_8 & +c_7 & -c_6 & +c_5 & +c_4 & -c_3 & +c_2 & +c_1 & +c_{16} & +c_{15} \\ +c_9 & +c_{10} & +c_{11} & +c_{12} & +c_{13} & +c_{14} & +c_{15} & +c_{16} & +c_1 & +c_2 \\ +c_{10} & +c_9 & +c_{12} & +c_{11} & +c_{14} & +c_{13} & +c_{16} & +c_{15} & +c_2 & +c_1 \end{matrix}$$

$$H(Y)=\begin{matrix}
y_1 & y_2 & y_3 & -y_4 & y_5 & -y_6 & -y_7 & y_8 & y_9 & y_{10} \\
y_2 & y_1 & -y_4 & y_3 & -y_6 & y_5 & -y_8 & y_7 & y_{10} & y_9 \\
y_3 & y_4 & y_1 & -y_2 & -y_7 & y_8 & y_5 & -y_6 & 0 & 0 \\
y_4 & y_3 & -y_2 & y_1 & -y_8 & y_7 & -y_6 & y_5 & 0 & 0 \\
y_5 & y_6 & y_7 & -y_8 & y_1 & -y_2 & -y_3 & y_4 & 0 & 0 \\
y_6 & y_5 & y_8 & -y_7 & -y_2 & y_1 & y_4 & -y_3 & 0 & 0 \\
y_7 & -y_8 & y_5 & y_6 & -y_3 & -y_4 & y_1 & y_2 & 0 & 0 \\
y_8 & -y_7 & y_6 & y_5 & -y_4 & -y_3 & y_2 & y_1 & 0 & 0 \\
y_9 & y_{10} & 0 & 0 & 0 & 0 & 0 & 0 & y_1 & y_2 \\
y_{10} & y_9 & 0 & 0 & 0 & 0 & 0 & 0 & y_2 & y_1 \\
0 & 0 & y_9 & y_{10} & 0 & 0 & 0 & 0 & y_3 & y_4 \\
0 & 0 & y_{10} & y_9 & 0 & 0 & 0 & 0 & y_4 & y_3 \\
0 & 0 & 0 & 0 & y_9 & y_{10} & 0 & 0 & y_5 & y_6 \\
0 & 0 & 0 & 0 & y_{10} & y_9 & 0 & 0 & y_6 & y_5 \\
0 & 0 & 0 & 0 & y_{10} & y_9 & 0 & 0 & y_6 & y_5 \\
0 & 0 & 0 & 0 & 0 & 0 & y_9 & y_{10} & y_7 & y_8 \\
0 & 0 & 0 & 0 & 0 & 0 & y_{10} & y_9 & y_8 & y_7
\end{matrix}\,.$$

This 'non-euclidean' factorization seems to be new.

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